Multiple dimensions:

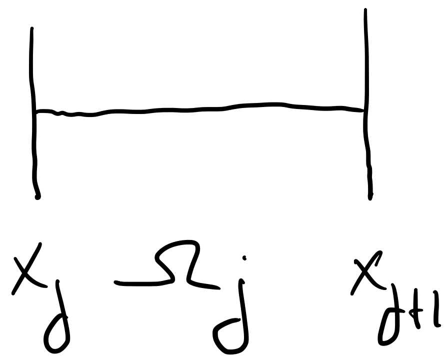
The one dimensional DG scheme for $u_t + a u_x = 0$ is

$$\int_{\Omega_j} \phi (u_t + a u_x) \, dx =$$

weak form:

Strong form (no requirement on ϕ)

$$\int_{\Omega_j} \phi u_t - a u \phi_x \, dx = -\left[a u \phi \right]_{x_j}^{x_{j+1}}$$



what are the outward pointing normals?

So we have

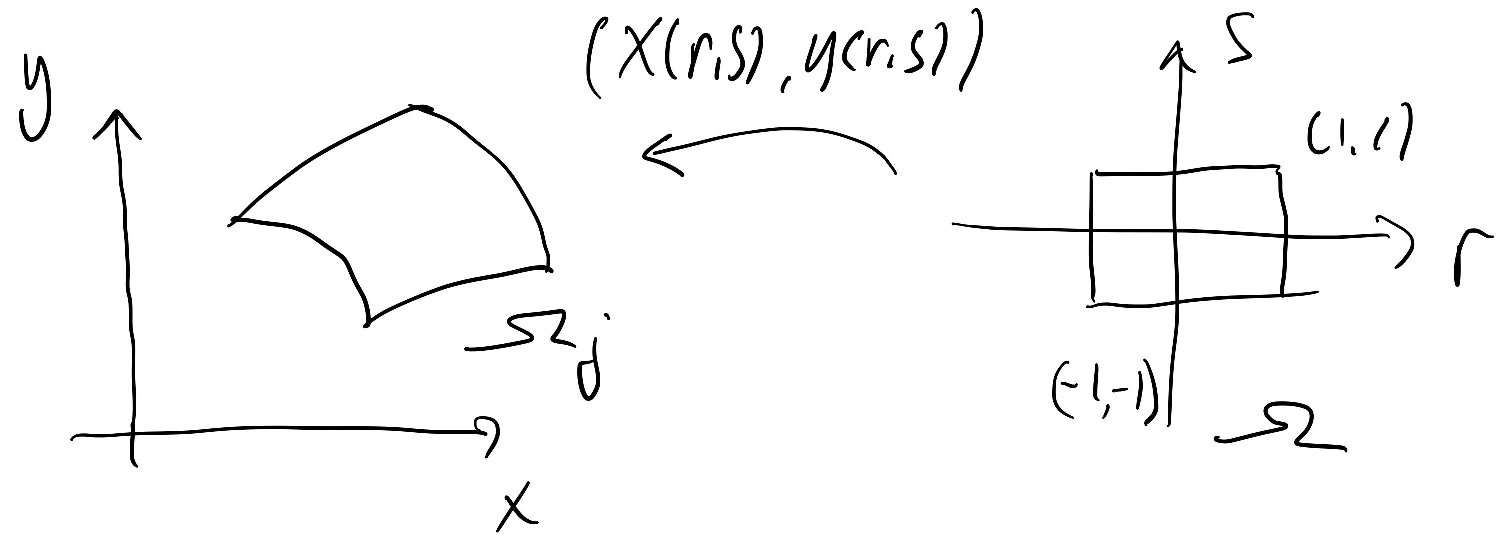
$$\int_{\Omega_j} \phi u_t - a u \phi_x \, dx = - \int_{\partial \Omega_j}$$

In multiple (2D say) dimension we have the advection equation

$$u_t(\bar{x},t) + \nabla \cdot f(u(\bar{x},t), \bar{x}, t) = 0$$

We will expand

$$u(\bar{x},t) = \sum_{k=0}^{q} \sum_{l=0}^{q} \hat{u}_{kl}(t)\, \phi_k(r)\, \phi_l(s)$$



The schemes are derived in the same way but now we use higher dimension versions of fundamental thm of calculus

Gauss / Divergence thm.

Divergence theorem

$$\underbrace{\iiint}_{n \text{ dims}} \nabla \cdot F \, d\Omega = \underbrace{\iint}_{n-1 \text{ dims}} \bar{n} \cdot \bar{f} \, dS$$

or in our case

$$\iint_{\Omega_j} \phi \nabla \cdot f \, d\Omega = \int_{\partial \Omega_j} \phi \, \bar{n} \cdot \bar{f} \, ds - \iint_{\Omega_j} \nabla \phi \cdot \bar{f} \, d\Omega$$

Results in the schemes

Weak

$$\iint_{\Omega_j} \phi u_t - \nabla\phi \cdot \bar{F} \, d\Omega_j = -\int_{\partial\Omega_j} (\bar{n} \cdot \bar{F})^* \phi \, ds$$

Strong

$$\iint_{\Omega_j} \phi u_t + \phi \nabla \cdot \bar{F} \, d\Omega_j = \int_{\partial\Omega_j} \bar{n} \cdot (\bar{F} - \bar{F}^*) \phi \, ds$$

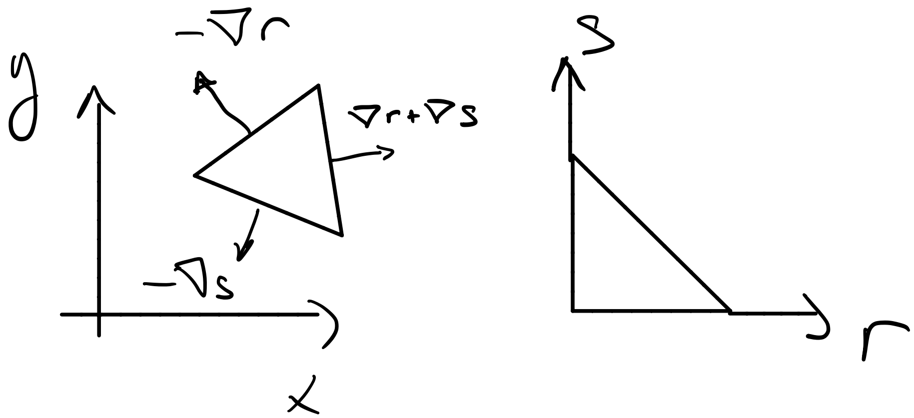
$$\iint_{\Omega_j} \phi u_t - \nabla\phi \cdot \bar{F} \, d\Omega_j = -\int_{\partial\Omega_j} (\bar{n}\cdot\bar{F})^* \phi \, ds \quad ,$$

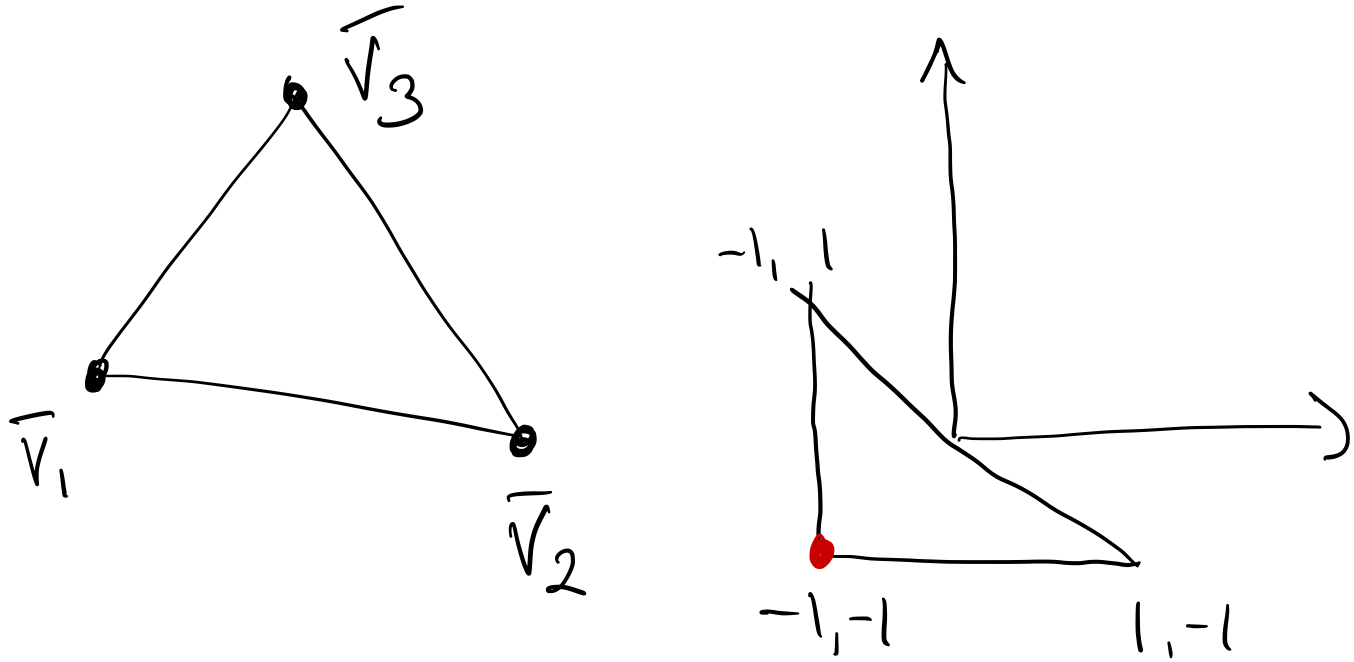
$$\bar{f}^*(a,b) = \frac{\bar{f}(a)+\bar{f}(b)}{2} + \frac{C}{2}\hat{n}(a-b)$$

$$C = \max_{u\in[a,b]} \left| n_x \frac{\partial f_1}{\partial x} + n_y \frac{\partial f_2}{\partial y} \right|$$

Lax-Friedrich flux.

How to get from Ω_j to Ω ?





Find a function of r and s that is zero at the top corner and at the lower right corner and one in the bottom left.

$$\bar{x} = -\frac{r+s}{2}\bar{v}_1 + \frac{r+1}{2}\bar{v}_2 + \frac{s+1}{2}\bar{v}_3$$

$$\iint_{\Omega_j} f(x,y)\,dx\,dy = \iint_{\Omega} f(x(r,s), y(r,s))\, J(r,s)\, dr\,ds$$

The Jacobian is the scaling of a infinitesimal element (a determinant)

How are dx, dy related to dr, ds?
Use chain rule.