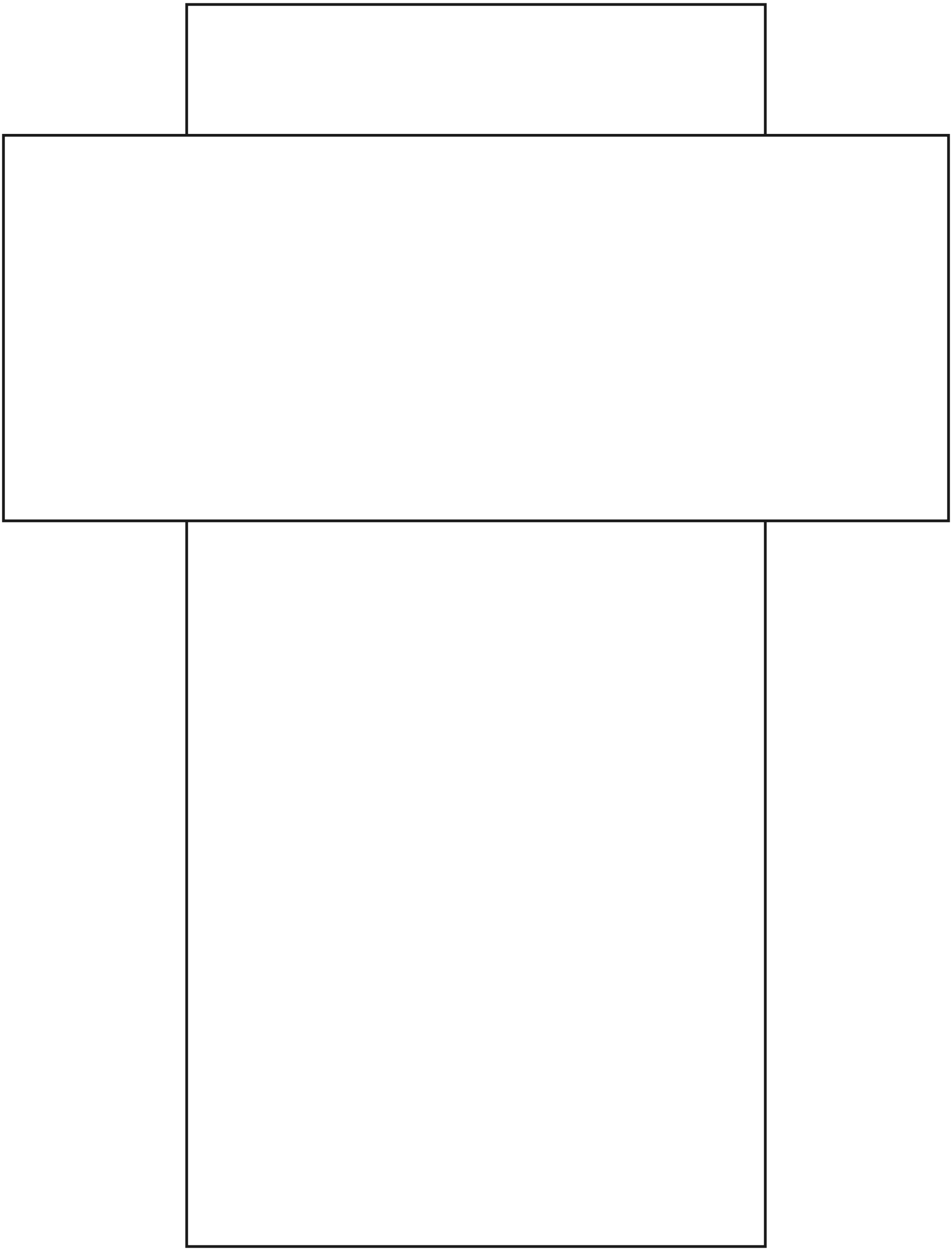


Today: Get an account at

rc.colorado.edu

1. Use your identikey and 2-factor auth
The smart phone app Duo is easy to use
and more convenient than getting a token.

Let me know when you are done

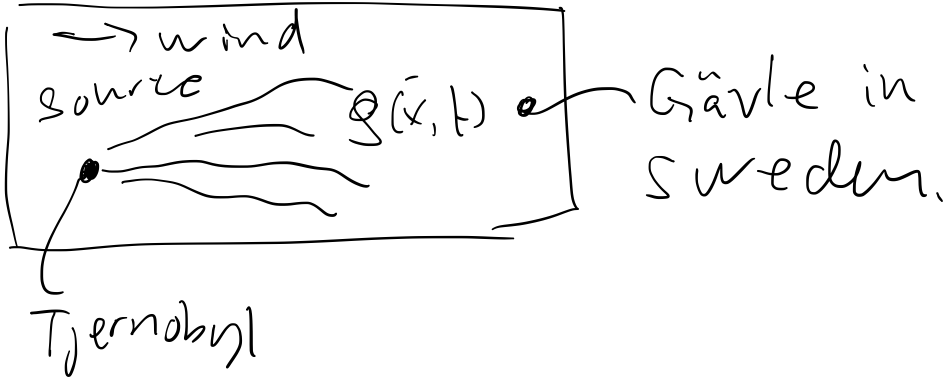
when everyone are done I will request an
allocation for the class.

Project: Group 1: Ferro Fluids Navier Stokes.

Group 2: Spread of polutant from $\delta(\bar{x}-\bar{x}_s)$

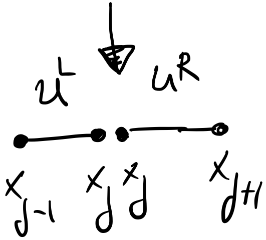
Group 3: Compressible Euler + Les type model.

2: Eq: $\rho_t + \nabla \cdot (\bar{u}\rho) = S(t)\,\delta(\bar{x}-\bar{x}_s)$



DG Systems and Multiple dimensions this week

$u^* = \beta u^L + (1-\beta) u^R$



Recall the scalar problem $u_t + a u_x = 0$

$$\int_{x_j}^{x_{j+1}} \phi u_t - a\phi_x u \, dx = -a\phi u^*(x_{j+1}) + a\phi u^*(x_j)$$

$x_j = \cancel{X}R$ ("just to the Right of x_j")

add on

$$\int_{x_{j-1}}^{x_j = x_L} \phi u_t - a\phi_x u \, dx = -a\phi u^*(x_j) + a\phi u^*(x_{j-1})$$

("just to the left of x_j")

$$\int_{x_R} \phi u^R_t - a\phi_x u^R \, dx = -a\phi u^*(x_{j+1}) + a\phi u^*(x_j)$$

Ignore pink terms.

$$\int^{x_L} \phi u^L_t - a\phi_x u^L \, dx = -a\phi u^*(x_j) + a\phi u^*(x_{j-1})$$

$$\phi \to u \quad \text{and} \quad \int_a^b u_x u \, dx = \frac{1}{2}u^2(b) - \frac{1}{2}u^2(a)$$

$$\int u u_t \, dx = a u^R u^* - a u^L u^* - \frac{1}{2} a u^R u^R + \frac{1}{2} a u^L u^L$$

(Ignored "outer terms")

$$= a\left((u^R - u^L)u^* + \frac{u^L u^L}{2} - \frac{u^R u^R}{2}\right)$$

$$a\left((u^R-u^L)u^* + \frac{u^Lu^L}{2} - \frac{u^Ru^R}{2}\right) = \left\{ u^* = \beta u^L + (1-\beta)u^R \right\}$$

$$a\left((u^R-u^L)\beta u^L + (u^R-u^L)(1-\beta)u^R + \frac{u^Lu^L}{2} - \frac{u^Ru^R}{2}\right) =$$

$$a\left(u^Lu^L\left(\frac{1}{2}-\beta\right) + u^Ru^R\left(\underbrace{-\frac{1}{2}+1-\beta}_{1/2-\beta}\right) + u^Lu^R\left(\beta+\beta-1\right)\right) =$$

$$a\left(u^L-u^R\right)^2\left(\frac{1}{2}-\beta\right)$$ So if $a<0$, $\beta<\frac{1}{2}$ $<0.$

System 1 D   How to choose flux?

Start with $\rho_t + u_x = 0$, ρ is density and u is velocity perturbations of a quiscent fluid.

$u_t + \rho_x = 0$

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \bar{w}_t + A\bar{w}_x = \bar{0}$$

what does upwind mean now? in 2 variables.

Note:

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigen vectors values

$$\lambda_1 = 1, \ \bar{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$\lambda_2 = -1, \ \bar{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

let $S = (\bar{v}_1, \bar{v}_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and introduce

$$\begin{pmatrix} q \\ r \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \varrho \\ u \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \varrho + u \\ \varrho - u \end{pmatrix}$$

characteristic variables

$$\frac{\partial}{\partial t}\begin{pmatrix} \varrho \\ u \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \varrho \\ u \end{pmatrix} = 0 \iff \frac{\partial}{\partial t} S^{-1}\begin{pmatrix} q \\ r \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1}\frac{\partial}{\partial x}\begin{pmatrix} q \\ r \end{pmatrix} = 0$$

multiply with S

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ r \end{pmatrix} + \underbrace{S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1}}_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\frac{\partial}{\partial x}\begin{pmatrix} q \\ r \end{pmatrix} = 0 \iff \begin{matrix} q_t + q_x = 0, \\ r_t - r_x = 0. \end{matrix}$$

$\frac{Dq}{Dt} = 0$

S diagonalizes A.

We have thus reduced the problem to two scalar eq and can write down a DG scheme

$$\int_{x_j}^{x_{j+1}} \phi q_t - \phi_x q \, dx = -\left[\phi q^*\right]_{x_j}^{x_{j+1}}, \qquad q^* = \beta_1 q^+ + (1-\beta_1) q^-,$$

$$\int_{x_j}^{x_{j+1}} \psi r_t + \psi_x r \, dx = +\left[\psi r^*\right]_{x_j}^{x_{j+1}}, \qquad r^* = \beta_2 r^+ + (1-\beta_2) r^-,$$

Choose β_1 and β_2 by the same analysis as before.

This approach works in 1D and one can convert to ϱ, u whenever needed.

In 2D the characteristics will be different and we cannot (typically) work in characteristic variables.

Let's try to work directly with

$$\bar{w}_t + A\bar{w}_x = 0.$$

Suppose that all eigenvalues of A are real and that there is a full set of eigenvectors

(The problem is hyperbolic)

Also assume that $\max\{|\lambda|\} \leq \lambda_{LF}$ then

We can use the scheme:

$$\int_{x_j}^{x_{j+1}} \bar{\phi}^T \bar{w}_t \, dx + \int_{x_j}^{x_{j+1}} \bar{\phi}^T A \bar{w}_x \, dx = \left[\bar{\phi}^T \left(A(\bar{w} - \bar{w}^*) \right) \right]_{x_j}^{x_{j+1}}$$

LAX FRIEDRISH

$$\text{where } A\bar{w}^* = A\left(\frac{\bar{w}^+ + \bar{w}^-}{2} \right) + \frac{\lambda_{LF}}{2} \left(n^- \bar{w}^- + n^+ \bar{w}^+ \right)$$

$$= A \{\{ \bar{w} \}\} + \frac{\lambda_{LF}}{2} [[\bar{w}]]$$

Here "$-$" means on my elem and "$+$" on neighbour.

Ex: $\bar{W} = \begin{pmatrix} \rho \\ u \end{pmatrix}$, $\bar{\Phi} \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ 0 \end{pmatrix} \ldots \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \ldots \right\} \ldots \begin{pmatrix} \phi \\ \psi \end{pmatrix}$

First set,

$$\int_{x_j}^{x_{j+1}} (\phi, 0) \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} dx + \int_{x_j}^{x_{j+1}} (\phi, 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} dx = \left[(\phi, 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} \rho \\ u \end{pmatrix} - \begin{pmatrix} \rho^* \\ u^* \end{pmatrix} \right] \right]_{x_j}^{x_{j+1}}$$

$$\int_{x_j}^{x_{j+1}} \phi \rho_t + \phi u_x \, dx = \left[\phi (u - u^*) \right]_{x_j}^{x_{j+1}}$$

Same for other eq.

$$\int_{x_j}^{x_{j+1}} \psi u_t + \psi \rho_x \, dx = \left[\psi (\rho - \rho^*) \right]_{x_j}^{x_{j+1}}$$

$$\int_{x_j}^{x_{j+1}} \phi \rho_t + \cancel{\phi u_x} \, dx = \left[\phi \left(\frac{u}{2} - u^* \right) \right]_{x_j}^{x_{j+1}}$$

$$\int_{x_j}^{x_{j+1}} \psi u_t + \cancel{\psi \rho_x} \, dx = \left[\psi \left(\frac{\rho}{2} - \rho^* \right) \right]_{x_j}^{x_{j+1}}$$

Here I choose

$n^- = -1$

Adding up contributions we find on element j

$$\rho^+\left(\frac{u^+}{2} - u^*\right) - \rho^-\left(\frac{u^-}{2} - u^*\right) + u^+\left(\frac{\rho^+}{2} - \rho^*\right) - u^-\left(\frac{\rho^-}{2} - \rho^*\right)$$

$$= \rho^+ u^+ - \rho^- u^- + u^*(\rho^- - \rho^+) + \rho^*(u^- - u^+)$$

$$\rho^+u^+ - \rho^-u^- + u^*(\rho^- - \rho^+) + \rho^*(u^- - u^+)$$

$$\rho^* = \frac{1}{2}(\rho^+ + \rho^-) + \frac{\lambda_{max}}{2}(n^-u^- + n^+u^+) = \frac{1}{2}(\rho^+ + \rho^- + u^+ - u^-)$$

$$u^* = \frac{1}{2}(u^+ + u^-) + \frac{\lambda_{max}}{2}(n^-\rho^- + n^+\rho^+) = \frac{1}{2}(u^+ + u^- + \rho^+ - \rho^-)$$

$$\rho^+u^+ - \rho^-u^- + \frac{1}{2}(\rho^- - \rho^+)\left[(\rho^+ - \rho^-) + u^+ + u^-\right] + \frac{1}{2}(u^- - u^+)\left[(u^+ - u^-) + \rho^+ + \rho^-\right]$$

$$= \rho^+u^+ - \rho^-u^- + \frac{1}{2}\rho^-u^+ + \frac{1}{2}\rho^-u^- - \frac{\rho^+}{2}u^+ - \frac{\rho^+u^-}{2} - \frac{1}{2}(\rho^- - \rho^+)^2$$

$$+ \frac{1}{2}u^-\rho^+ + \frac{1}{2}u^-\rho^- - \frac{1}{2}u^+\rho^+ - \frac{1}{2}u^+\rho^- - \frac{1}{2}(u^- - u^+)^2$$

Dissipation due to upwinding

Higher order equations. (Basics)

Consider the heat eq.

$$u_t = u_{xx}, \qquad -\pi \leq x \leq \pi, \; t \geq 0$$

$$u(x,0) = \sin(x)$$

has solution $u(x,t) = e^{-t} \sin(x)$.

To make this look like a first order eq. we write $\frac{\partial}{\partial t} u = \frac{\partial}{\partial x} u_x$

$$\frac{\partial}{\partial t} u = \frac{\partial}{\partial x} u_x$$

. Introducing $q = u_x$

we have

$$\left.\begin{aligned} u_t &= q_x \\ q &= u_x \end{aligned}\right\}$$

$$\int \phi u_t - \phi q_x \, dx = -\left[\phi(q - q^*)\right] \qquad (1)$$

$$\int \psi q \, dx = \int \psi u_x \qquad (2)$$

We try to solve 2 first as this is a linear system of eq. then we compute u_t to evolve

question: This is a heat eq, how should we choose q^*?

The "natural choice" $q^* = \{\{q\}\}$ does not work!

this was very surprising and was a mystery for some time. The central flux on q^* by itself is really unstable but often looks stable (although there is <u>no</u> convergence)

what to do?

Another formulation

$$\int_{\Omega_j} \phi\, u_t\, dx = \int_{\Omega_j} \phi\, q_x\, dx - \left[\phi(q - q^*)\right]_{x_j}^{x_{j+1}}$$

$$\int_{\Omega_j} \psi\, q\, dx = \int_{\Omega_j} \psi\, u_x\, dx - \left[\psi(u - u^*)\right]_{x_j}^{x_{j+1}}$$

To get a local method we choose

$u^* = \frac{1}{2}(u^+ + u^-) \rightarrow$ Solve locally for q everywhere

then choose $q^* = \frac{1}{2}(q^+ + q^-)$ and compute u_t locally.

It turns out that this gives a stable method but a method that is only optimally convergent for every other degree (which is not really a big deal?)

An alternative is to choose so-called local DG (LDG) fluxes $q^* = q^+$, $u^* = u^-$

or $q^* = q^-$, $u^* = u^+$

this gives optimal convergence.

Alternating sides for u and q..

At boundaries one may enforce Dirichlet bc by outside states $u^+ = -u^-$, $q^+ = q^-$

or Neumann BC as $u^+ = u^-$, $q^+ = -q^-$

Show program/results for

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = (u_{xx} + u_{yy})\nu$$

Non linear terms.

A safe but slow way to implement a non linear term is

$$\int \phi u_t - \phi_x \left(\frac{u^2}{2}\right) dx = \left[\phi \left(\frac{u^2}{2}\right)^* \right]$$

Eval u on quadrature nodes, square, integrate.

Note that deg is $3 \cdot \deg(\phi)$

Treated with Lax-Friedrich flux

$$\left(\frac{u^2}{2}\right)^* = \left(\frac{(u^+)^2}{2} + \frac{(u^-)^2}{2}\right)\frac{1}{2} + \frac{|u|_\infty}{2} [[u]]$$