Num. PDE 1

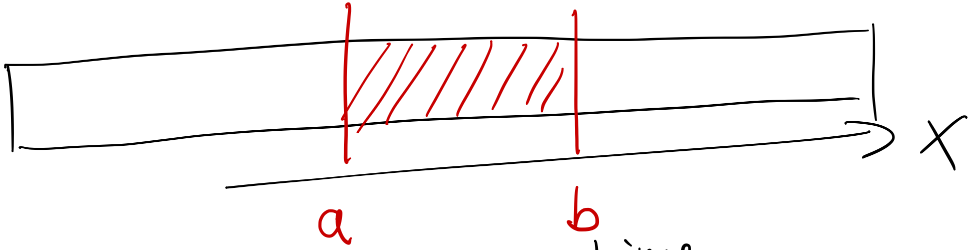
Finite volume + ERROR est.
Convergence / consistency

Next few weeks numerical solution of PDE.

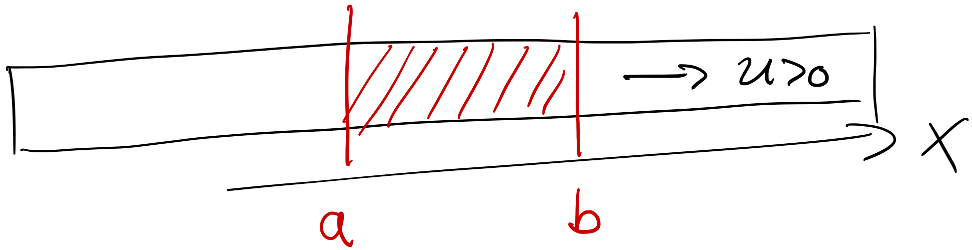
- Basic Schemes FV then DG
- Stability
- Consistency
- Convergence

Notes based on Tim Warburton's Class notes from Math 578 at UNM.

Consider a fluid in a pipe with density $\rho(x,t)$ flowing with a velocity $u(x,t)$



rate of change in time inside the red part is computed by comparing how much fluid flows out from each end



Rate of change is

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = -u(b,t)\,\rho(b,t) + u(a,t)\,\rho(a,t)$$

Rate of change of total mass in section

Flux Out of section

Flux into section

Note that:

$$-u(b,t)\rho(b,t) + u(a,t)\rho(a,t) =$$

$$= \frac{d}{dx}\int_a^b u(x,t)\rho(x,t)\,dx \quad \text{so } \nabla$$

$$\int_a^b \frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\left(u(x,t)\rho(x,t)\right) dx = 0$$

holds for any a, b $\Longrightarrow$

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}(\rho u) = 0$$

A particularly simple case is when u is constant $u(x,t) = \bar{u}$, then

$$\frac{\partial}{\partial t}\rho + u\frac{\partial}{\partial x}\rho = 0.$$

We can solve this by change of variables

$$\hat{t} = t, \quad \hat{x} = x - \bar{u}t, \quad \Rightarrow \quad \frac{\partial}{\partial t} = \frac{\partial \hat{t}}{\partial t}\cdot\frac{\partial}{\partial \hat{t}} + \frac{\partial \hat{x}}{\partial t}\cdot\frac{\partial}{\partial \hat{x}}$$

$$= \frac{\partial}{\partial \hat{t}} - \bar{u}\frac{\partial}{\partial \hat{x}}$$

Also: $\dfrac{\partial}{\partial x} = \dfrac{\partial \hat{t}}{\partial x}\dfrac{\partial}{\partial \hat{t}} + \dfrac{\partial \hat{x}}{\partial x}\dfrac{\partial}{\partial \hat{x}}$

$$= \frac{\partial}{\partial \hat{x}}$$

So:

$$0 = \frac{\partial}{\partial t}\rho + \bar{u}\frac{\partial}{\partial x}\rho = \left(\frac{\partial}{\partial \hat{t}} - \bar{u}\frac{\partial}{\partial \hat{x}}\right)\rho + \bar{u}\frac{\partial}{\partial \hat{x}}\rho$$

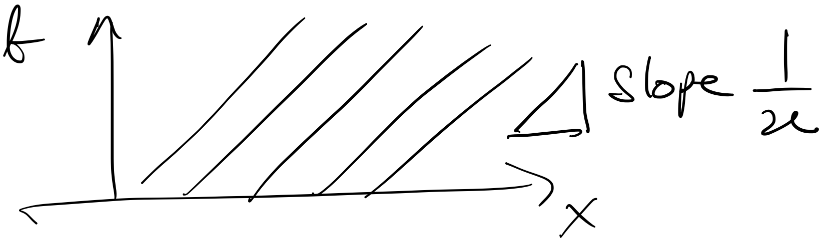
$$= \frac{\partial}{\partial \hat{t}}\rho$$

$$\boxed{\frac{\partial}{\partial \hat{t}}\rho = 0}$$

EASY to solve

$$\rho(x, t) = \rho_0(\hat{x}) = \rho(x - \bar{u}t)$$

The solution is translated in space as time increases



A finite volume solver from:

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = -\bar{u}\,\rho(b,t) + \bar{u}\,\rho(a,t)$$

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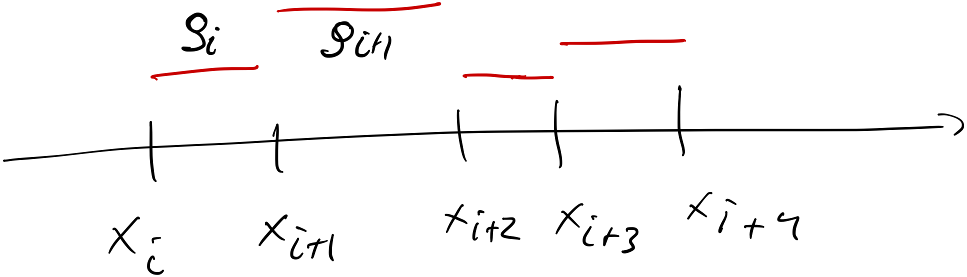
divide the real line into elements



In each element approx. ρ by a constant

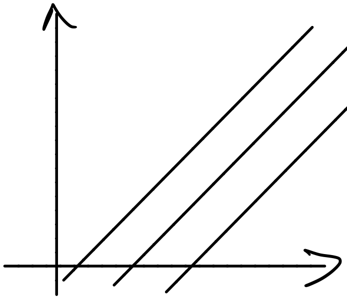
ρ_i

$$\frac{d}{dt}\int_{x_i}^{x_{i+1}} \rho(x,t)\,dx = -\bar{u}\,\rho(x_{i+1},t) + \bar{u}\,\rho(x_i,t)$$

Note that $\displaystyle\int_{x_i}^{x_{i+1}} \rho(x,t)\,dx = \bar{\rho}_i \cdot \Delta x$ so we can approx.

$$\frac{1}{\Delta t}\left(\bar{\rho}_i(t_{n+1}) - \bar{\rho}_i(t_n)\right)\cdot \Delta x = -\bar{u}\,\underline{\rho(x_{i+1},t_n)} + \bar{u}\,\underline{\rho(x_i,t_n)}$$

How to recover values at x_i, x_{i+1} from averages?



Use the upwind values to evaluate the values at the boundaries. Here upwind is to the left.

$$\frac{1}{\Delta t}\left(\bar{\rho}_i(t_{n+1}) - \bar{\rho}_i(t_n)\right)\cdot \Delta x = -\bar{u}\rho(x_{i+1}, t_n) + \bar{u}\rho(x_i, t_n)$$

$$= -\bar{u}\,\bar{\rho}_i(t_n) + \bar{u}\,\bar{\rho}_{i-1}(t_n)$$

Simplify: $\bar{\rho}_i^{n+1} = \left(1 - \bar{u}\frac{\Delta t}{\Delta x}\right)\bar{\rho}_i^n + \left(\bar{u}\frac{\Delta t}{\Delta x}\right)\bar{\rho}_{i-1}^n$

$$\bar{\rho}_i^{n+1} = (1-\lambda)\bar{\rho}_i^n + \lambda\bar{\rho}_{i-1} \qquad i = 1, 2 \ldots$$

ρ_0^n = Boundary cond.

We now have a reasonable scheme but how good is it?

Let q be the exact solution and

$$\overline{q}_i^n = \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} q(x, n\cdot\Delta t)\,dx = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} q(x, n\Delta t)\,dx$$

q satisfies
$$\frac{d}{dt}\int_{x_i}^{x_{i+1}} q(x,t)\,dx = \frac{d}{dt}\left(\Delta x\,\overline{q}_i\right) = -\overline{u}\,q(x_{i+1},t) + \overline{u}\,q(x_i,t)$$

We want to estimate the error, i.e the difference between the numerical and exact solution at time $T = n\Delta t$,

- we thus seek the error $E_i^n = |\bar{q}_i^n - \bar{g}_i^n|$, $n = \frac{T}{\Delta t}$.
- To take limits we insist that Δt and Δx are coupled by $\Delta t = C \cdot \Delta x$ for some fixed C.
- If we take $\Delta t \to 0 \implies E^n = \mathcal{O}(\Delta t^s)$ we say that the method is s:th order accurate.

Let $\|E\|_p = \left(\Delta t \sum |E_i|^p \right)^{1/p}$.

The scheme is said to be convergent in the norm $\|\cdot\|$ at time T if

$$\lim_{\Delta t \to 0} \|E^n\| = 0$$

$T = n\Delta t$

Plan of attack, Relate errors at subsequent timesteps and find a bound at T.

We get to use the method

We start by estimating the local truncation error.

The local truncation error is the error commited in <u>one timestep</u> assuming we start with <u>exact initial data</u>

$\therefore \quad \bar{\rho}_i^n = \bar{q}_i^n \,, \quad$ and $\quad \bar{\rho}_i^{n+1} = \left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)\bar{q}_i^n + \frac{\Delta t}{\Delta x}\bar{u}\,\bar{q}_{i-1}^n$

$$LTE = R_i^n \equiv \frac{1}{\Delta t}\left(\bar{\rho}_i^{n+1} - \bar{q}_i^{n+1}\right)$$

<u>TAYLOR</u>

$$\bar{q}^{n}_{i-1} = \bar{q}^{n}_{i} - \Delta x \overline{\left(\frac{\partial q}{\partial x}\right)} + \frac{\Delta x^2}{2} \overline{\left(\frac{\partial^2 q}{\partial x^2}\right)} + \mathcal{O}(\Delta x^3)$$

$$\bar{q}^{n+1}_{i} = \bar{q}^{n}_{i} + \Delta t \overline{\left(\frac{\partial q}{\partial t}\right)} + \frac{\Delta t^2}{2} \overline{\left(\frac{\partial^2 q}{\partial t^2}\right)} + \mathcal{O}(\Delta t^3)$$

$$R^n_i = \frac{1}{\Delta t}\left(\left(1 - \frac{\Delta t}{\Delta x}\bar{u}\right)\bar{q}^n_i + \frac{\Delta t}{\Delta x}\bar{u}\,\bar{q}^n_{i-1} - \bar{q}^{n+1}_i\right)$$

Simplify

DO IT!

$$= -\left(\overline{\frac{\partial q}{\partial t}} + \bar{u}\overline{\frac{\partial q}{\partial x}}\right) - \left\{ \frac{\Delta t}{2}\overline{\left(\frac{\partial^2 q}{\partial t^2}\right)} - \frac{\bar{u}\,\Delta x}{2}\overline{\left(\frac{\partial^2 q}{\partial x^2}\right)} + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)\right.$$

Simplify more: $\overline{\frac{\partial q}{\partial t}} + \bar{u}\,\overline{\frac{\partial q}{\partial x}} = 0$, $\overline{\frac{\partial^2 q}{\partial t^2}} = \bar{u}^2\,\overline{\frac{\partial^2 u}{\partial x^2}}$

$$R_i^n = \frac{\bar{u}\,\Delta x}{2}\left(1 - \frac{\bar{u}\,\Delta t}{\Delta x}\right)\overline{\frac{\partial^2 q}{\partial x^2}} + \mathcal{O}(\Delta x^2),$$

Note that $R_i^n \to 0$ as $\Delta t \to 0$

assuming $\Delta t = C \cdot \Delta x$

When the local truncation error $\to 0$ as $\Delta t \to 0$ we say that the method is consistent

The local truncation error predicts that we will make an Δt error every timestep. Let's be a bit more precise

DEF $\bar{\rho}_i^n = \bar{q}_i^n + E_i^n$ and write the scheme as

$$\bar{\rho}^{n+1} = N \bar{\rho}^n .$$

We have $\bar{E}^{n+1} = N(\underbrace{\bar{q}^n + E^n}_{\bar{\rho}^n}) - \bar{q}^{n+1} = N(\bar{q}^n + \bar{E}^n) - N(\bar{q}^n) + N(\bar{q}^n) - \bar{q}^{n+1}$

$$= \underbrace{N(\bar{q}^n + \bar{E}^n) - N(\bar{q}^n)}_{\text{Propagation of previous ERROR}} + \Delta t R^n$$

TRUNCATION ERROR

Now suppose that for some norm N is a contraction.

$$\| N(p) - N(q) \| \leq \| p - q \|$$

$$E^{n+1} = N(\bar{q}^n + E^n) - N(\bar{q}^n) + \Delta t\, R^n$$, taking norms + Δ ineq.

$$\|E^{n+1}\| \leq \| N(\bar{q}^n + E^n) - N(\bar{q}^n)\| + \Delta t \|R^n\|$$

$$\leq \| \bar{q}^n + E^n - \bar{q}^n \| + \Delta t \|R^n\| \leq \|E^n\| + \Delta t \|R^n\|$$

$$\leq \|E^{n-1}\| + \Delta t \|R^{n-1}\| + \Delta t \|R^n\| \quad \{\text{many times}\}$$

$$\leq \|E^0\| + \Delta t \sum_{m=1}^{n} \|R^m\|$$

In other words the error at time T is

$$\|E^{n+1}\| \leq \|E^{0}\| + \Delta t \sum_{m=1}^{n} \|R^{m}\| \leq \|E^{0}\| + T \max_{m=1,\dots,n} \|R^{m}\|$$

So IF the method is consistent (and q is smooth)

$$\|E^{n+1}\| \leq \|E^{0}\| + T\,\mathcal{O}(\Delta x)$$

Initial data $\to 0$ as $\Delta x \to 0$

Homework: Prove that

$$\bar{\rho}_i^{n+1} = (1-\lambda)\,\bar{\rho}_i^{n} + \lambda\,\bar{\rho}_{i-1}^{n}$$ is a contraction

in $\|\cdot\|_1$ when $\bar{\rho}_0^{n} = 0$.

DG: Consistency - Stability - Convergence

We solved $\frac{d}{dt}\left(\Delta x\, \bar{q}_i(t)\right) = -\bar{u}\, q(x_{i+1}, t) + \bar{u}\, q(x_i, t)$

$$\simeq -\bar{u}\bar{q}_i + \bar{u}\bar{q}_{i-1}$$

upwind approx.

Our first DG scheme will generalize this so that we can exploit approximation by high degree polynomials

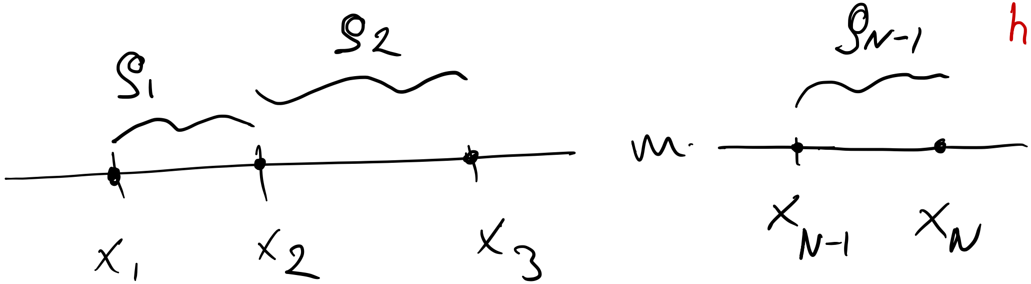
Again divide the pipe into segments separated by $x_1, x_2, x_3 \dots\dots x_N$.

On each segment $x_i \leq x \leq x_{i+1}$ we solve a weak form of the advection eq.

space of polynomials of deg p

Find $\rho_i^0 \in P^p(x_i, x_{i+1})$ such that $\forall v \in P^p(x_i, x_{i+1})$

$$\int_{x_i}^{x_{i+1}} v \left(\frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i)\, \bar{u} \left(\rho_{i-1}(x_i) - \rho_i^0(x_i) \right)$$



high deg poly

$$\int_{x_i}^{x_{i+1}} v \left(\frac{\partial \rho_i}{\partial t} + \bar{u} \frac{\partial \rho_i}{\partial x} \right) dx = v(x_i)\, \bar{u} \left(\rho_{i-1}(x_i) - \rho_i(x_i) \right)$$

Selecting $\rho_i \in P^0(x_i, x_{i+1})$ i.e. constants

Reduces this to finite vol.

What about stability?

The best part of dG is that this is trivial

- Find $\rho_i \in P^p(x_i, x_{i+1})$ such that $\forall\, v \in P^p(x_i, x_{i+1})$ ⟸ FOR ALL

$$\int_{x_i}^{x_{i+1}} v\left(\frac{\partial \rho_i}{\partial t} + \bar{u}\frac{\partial \rho_i}{\partial x}\right)dx = v(x_i)\left(\rho_{i-1}(x_i) - \rho_i(x_i)\right)$$

Select $v = \rho_i$!

$$\int_{x_i}^{x_{i+1}} \rho_i\left(\frac{\partial \rho_i}{\partial t} + \bar{u}\frac{\partial \rho_i}{\partial x}\right)dx = \rho_i(x_i)\left(\rho_{i-1}(x_i) - \rho_i(x_i)\right)$$

Simplify to get an energy estimate

$$\frac{d}{dt}\int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2}\,dx = -\bar{u}\left(\int_{x_i}^{x_{i+1}} \rho_i \frac{\partial \rho_i}{\partial x}\,dx\right) + \bar{u}\,\rho_i(x_i)\left(\rho_{i-1}(x_i) - \rho_i(x_i)\right)$$

IBP

$$= -\bar{u}\left[\frac{\rho_i^2}{2}\right]_{x_i}^{x_{i+1}} + \bar{u}\,\rho_i(x_i)\left(\rho_{i-1}(x_i) - \rho_i(x_i)\right)$$

$$= \bar{u}\left(-\frac{\rho_i^2(x_i)}{2} - \frac{\rho_i^2(x_{i+1})}{2}\right) + \bar{u}\,\rho_i(x_i)\,\rho_{i-1}(x_i)$$

$$\frac{d}{dt}\int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2}\,dx = \bar{u}\left(-\frac{\rho_i^2(x_i)}{2} - \frac{\rho_i^2(x_{i+1})}{2}\right) + \bar{u}\,\rho_i(x_i)\,\rho_{i-1}(x_i)$$

Sum over ELEMS

$$\sum_{i=1}^{N-1}: \quad \frac{d}{dt}\int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2}\,dx = \bar{u}\left(-\frac{\rho_i^2(x_i)}{2} - \frac{\rho_i^2(x_{i+1})}{2}\right) + \bar{u}\,\rho_i(x_i)\,\rho_{i-1}(x_i)$$

$$\frac{d}{dt}\sum_{i=1}^{N-1}\left(\int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2}\,dx\right) = \sum_{i=1}^{N-1}\left[\bar{u}\left(-\frac{\rho_i^2(x_i)}{2} - \frac{\rho_i^2(x_{i+1})}{2}\right) + \bar{u}\,\rho_i(x_i)\,\rho_{i-1}(x_i)\right]$$

$$\leq \sum_{i=1}^{N-1}\left[\bar{u}\left(\frac{-\rho_i^2(x_i) - \rho_i^2(x_{i+1})}{2}\right) + \bar{u}\left(\frac{\rho_i^2(x_i) + \rho_{i-1}^2(x_i)}{2}\right)\right]$$

$$\leq \frac{\bar{u}}{2}\sum_{i=1}^{N-1}\left[-\rho_i^2(x_{i+1}) + \rho_{i-1}^2(x_i)\right]$$

$$\leq \frac{\bar{u}}{2}\left(\rho_0^2(x_1) - \rho_{N-1}^2(x_n)\right)$$

Note that we left time continuous. If time derivatives are discretized exactly we have that the energy of the numerical solution is controled by the boundary condition $\rho_0(x_i)$

denoting $\|\rho\|_2^2 = \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} \frac{\rho_i^2}{2} dx$ we find

$$\frac{d}{dt} \frac{\|\rho\|_2^2}{2} \leq \frac{\bar{u}}{2} \left(\rho_0^2(x_i) - \rho_{N-1}^2(x_N) \right)$$

or $$\frac{d}{dt} \|\rho\|_2 \leq \frac{\bar{u}}{2\|\rho\|_2} \left(\rho_0^2(x_i) - \rho_{N-1}^2(x_N) \right)$$

Now we can try to bound the error in the same way.

Eg: Let ρ and σ be two numerical solutions that are close at the initial time, then trivially (same BC)

$$\frac{d}{dt}\|\rho-\sigma\|_2 \leq \frac{-\bar{u}\left(\rho_{N-1}(x_N) - \sigma_{N-1}(x_N)\right)^2}{2\|\rho-\sigma\|_2}$$

$\leq 0 \rightarrow$ the solutions get closer with time.

1. Let q be the exact solution
2. Assume that we have a projection operator

$$\Pi^p : H^1(x_1, x_N) \to \bigcup_{i=1}^{N-1} P^p(x_i, x_{i+1})$$

We work with 3 - equations

TRUNCATION
LOC.

(1) $$\int v\, \Pi^p \left(\frac{\partial q_i}{\partial t} + \bar{u} \frac{\partial q_i}{\partial x} \right) dx = 0$$

(2) $$\int v \left(\frac{\partial \Pi^p q_i}{\partial t} + \bar{u} \frac{\partial}{\partial x} \Pi^p q_i \right) dx = v(x_i) \left(\Pi^p q_{i-1}(x_i) - \Pi^p q_i(x_i) \right) + \int v R_i$$

(3) $$\int v \left(\frac{\partial \varrho_i}{\partial t} + \bar{u} \frac{\partial \varrho_i}{\partial x} \right) dx = v(x_i) \left(\varrho_{i-1}(x_i) - \varrho_i(x_i) \right)$$

1. Project PDE.
2. Truncation error.
3. The method.

Assume that projection and $\frac{\partial}{\partial t}$ commute and that q is smooth.

1 + 2 gives

$$\int_{x_i}^{x_{i+1}} v R_i \, dx = \int v \left(\bar{u} \frac{\partial \Pi^p q_i}{\partial x} - \bar{u} \Pi^p \frac{\partial q_i}{\partial x} \right) dx - v(x_i) \Big[\Pi^p q_{i-1}(x_i) - q_{i-1}(x_i) + q_i(x_i) - \Pi^p q_i(x_i) \Big]$$

Now use $v = R_i \in P^p$ + C-S + Δ ineq.

$$\| R \|_2 \leq \underbrace{\sum_{i=1}^{N-1} \left\| \bar{u} \frac{\partial \Pi^p q_i}{\partial x} - \bar{u} \Pi^p \frac{\partial q_i}{\partial x} \right\|_2}_{C h^s \left\| \frac{d^p q}{d x^p} \right\|_2} + \sum_{i=1}^{N-1} \Big(\left| \Pi^p q_{i-1}(x_i) - q_i(x_i) \right| + \left| \Pi^p q_i(x_i) - q_i(x) \right| \Big)$$

We have an estimate for truncation error

Now we estimate $\| \Pi^p q - \varrho \|_2$ by ③ − ②.

② $\displaystyle\int v \left(\frac{\partial}{\partial t} \Pi^p q_i + \bar{u} \frac{\partial \Pi^p q_i}{\partial x} \right) dx = v(x_i) \left(\Pi^p q_{i-1}(x_i) - \Pi^p q_i(x_i) \right) + \int_{x_i}^{x_{i+1}} v R_i$

③ $\displaystyle\int v \left(\frac{\partial \varrho_i}{\partial t} + \bar{u} \frac{\partial \varrho_i}{\partial x} \right) dx = v(x_i) \left(\varrho_{i-1}(x_i) - \varrho_i(x_i) \right)$

Subtract ③ from ② and set $v = (\Pi^p q_i - \varrho_i)$

$$\int_{x_i}^{x_{i+1}} (\Pi q_i - \varrho_i) \left(\frac{\partial \Pi q_i - \varrho_i}{\partial t} + \bar{u} \frac{\partial \Pi q_i - \varrho_i}{\partial x} \right) dx =$$

$$(\Pi q_i(x_i) - \varrho_i(x_i)) \left(\Pi q_{i-1}(x_i) - \varrho_{i-1}(x_i) - \Pi q_i(x_i) + \varrho_i(x_i) \right)$$

$$+ \int_{x_i}^{x_{i+1}} (\Pi q_i - \varrho_i) R_i$$

$$\boxed{\text{USE the stability Res} \quad \frac{d}{dt} \| \varrho - \sigma \|_2 \leq 0}$$

$$\frac{d}{dt}\frac{\|\Pi^p q - \varrho\|_2^2}{2} \leq \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} (\Pi^p q_i - \varrho_i)\, R_i\, dx$$

$$\leq \|\Pi^p q - \varrho\|_2 \, \|R\|_2$$

Use Estimate for $\|R\|$

$$\frac{d}{dt}\|\Pi^p q - \varrho\|_2 \leq \|R\| \leq C \cdot h^s \left\| \frac{d^p q}{dx^p} \right\|_2$$

The numerical sol. and the projection grows slowly apart in time.

We integrate in time:

$$\| \Pi^P q - \varrho \|_2 \leq \| \Pi^P q(t=0) - \varrho(t=0) \|_2 + C h^S T \max_{0 \leq \tau \leq T} \left\| \frac{d^P q}{dx^P}(\tau) \right\|_2$$

Finally we estimate error between exact and num. sol,

$$\| q - \varrho \|_2 = \| q - \Pi q + \Pi q - \varrho \|$$

$$\leq \| q - \Pi q \|_2$$

$$+ \| \Pi q(t=0) - \varrho(t=0) \|_2 + C h^S T \max_{0 \leq \tau \leq T} \left\| \frac{d^P q(\tau)}{dx^P} \right\|_2$$

First term is beyond our control

Second term suggests we approximate initial data accurately.

Third term leads to linear growth with T.

- We swept $Ch^s \left\| \frac{d^p q}{dx^p} \right\|_2$ under the rug
- In multi D the argument are the same
- In special cases one can cook up a better Π to get a sharper estimate (Does not effect the method though)

Next lecture we start thinking about multiple dimensions.